

A NEW q -SELBERG INTEGRAL, SCHUR FUNCTIONS, AND YOUNG BOOKS

JANG SOO KIM AND SOICHI OKADA

ABSTRACT. Recently, Kim and Oh expressed the Selberg integral in terms of the number of Young books which are a generalization of standard Young tableaux of shifted staircase shape. In this paper the generating function for Young books according to major index statistic is considered. It is shown that this generating function can be written as a Jackson integral which gives a new q -Selberg integral. It is also shown that the new q -Selberg integral has an expression in terms of Schur functions.

1. INTRODUCTION

For a positive integer n , and complex numbers α, β, γ with $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\operatorname{Re}(\gamma) > -\min\{1/n, \operatorname{Re}(\alpha)/(n-1), \operatorname{Re}(\beta)/(n-1)\}$, the Selberg integral $S_n(\alpha, \beta, \gamma)$ is defined by

$$(1) \quad S_n(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (1-x_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n \\ = \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\gamma) \Gamma(\beta + (j-1)\gamma) \Gamma(1+j\gamma)}{\Gamma(\alpha + \beta + (n+j-2)\gamma) \Gamma(1+\gamma)}.$$

The Selberg integral is a generalization of Euler's beta integral, which is $S_1(\alpha, \beta, 0)$. The formula above is due to Selberg [13]. There are many generalizations of the Selberg integral, see [2]. As a q -analogue of (1), Askey [1] conjectured that

$$(2) \quad \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (qx_i; q)_{\beta-1} \prod_{1 \leq i < j \leq n} x_i^{2k} \left(q^{1-k} \frac{x_j}{x_i}; q \right)_{2k} d_q x_1 \cdots d_q x_n \\ = q^{\alpha k \binom{n}{2} + 2k^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)k) \Gamma_q(\beta + (j-1)k) \Gamma_q(1+jk)}{\Gamma_q(\alpha + \beta + (n+j-2)k) \Gamma_q(1+k)},$$

where $(x; q)_s$ is the q -Pochhammer symbol and $\Gamma_q(x)$ is the q -Gamma function. The Askey conjecture (2) has been proved by Habsieger [3] and Kadell [5] independently.

When $\alpha-1 = r$, $\beta-1 = s$, and $2\gamma = m$ are nonnegative integers, the Selberg integral $S_n(\alpha, \beta, \gamma)$ has a combinatorial interpretation due to Stanley [14, Chapter 1, Exercise 11]. Kim and Oh [6] introduced a combinatorial object called “ $(n, \mathbf{r}, \mathbf{s})$ -Young books”, where $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{s} = (s_1, \dots, s_m)$ are (weak) compositions of r and s respectively, and showed that

$$(3) \quad \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{k=1}^m \left(\prod_{i=1}^n x_i^{r_k} (1-x_i)^{s_k} \prod_{1 \leq i < j \leq n} |x_j - x_i| \right) dx_1 \cdots dx_n \\ = \frac{1}{N!} \prod_{k=1}^m \frac{F(n+r_k+s_k)}{F(r_k)F(s_k)} |\text{YB}(n; \mathbf{r}, \mathbf{s})|,$$

where $N = m \binom{n}{2} + (r+s+1)n + \sum_{k=1}^m r_k s_k$, $F(l) = \prod_{i=1}^{l-1} i!$, and $\text{YB}(n; \mathbf{r}, \mathbf{s})$ is the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books. Notice that the left hand side of (3) is equal to $\frac{1}{n!} S_n(\alpha, \beta, \gamma)$ with $\alpha-1 = r$, $\beta-1 = s$, and $2\gamma = m$. Their proof of (3) uses Stanley's combinatorial interpretation for the

Selberg integral and Postnikov's result [12] on the generating function for the number of standard Young tableaux of shifted staircase shape.

In this paper we generalize (3) by considering a q -analogue for the Young books.

A Young book can be thought of as a linear extension of a poset. Thus a natural q -analogue of $|\text{YB}(n, \mathbf{r}, \mathbf{s})|$ is the generating function for $(n, \mathbf{r}, \mathbf{s})$ -Young books according to the major index statistic of linear extensions. Interestingly, this q -analogue is the q -Selberg integral which is obtained from (3) by simply replacing $(1-x)^s$ by $(qx; q)_s = \prod_{i=1}^s (1-q^i x)$ and the Riemann integral by the Jackson integral.

Theorem 1.1. *Let n and m be positive integers, and $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{s} = (s_1, \dots, s_m)$ compositions of r and s respectively. Then, for $0 < q < 1$, we have*

$$(4) \quad \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{k=1}^m \left(\prod_{i=1}^n x_i^{r_k} (qx_i; q)_{s_k} \prod_{1 \leq i < j \leq n} |x_j - x_i| \right) d_q x_1 \cdots d_q x_n \\ = q^{(r+1)\binom{n}{2} + m\binom{n}{3}} \frac{1}{[N]_q!} \prod_{k=1}^m \frac{F_q(n + r_k + s_k)}{F_q(r_k) F_q(s_k)} \sum_{B \in \text{YB}(n; \mathbf{r}, \mathbf{s})} q^{\text{maj}(B)},$$

where $N = m\binom{n}{2} + (r + s + 1)n + \sum_{k=1}^m r_k s_k$, and $F_q(l) = \prod_{i=1}^{l-1} [i]_q!$.

As a part of the proof of Theorem 1.1, we show that the q -Selberg integral in Theorem 1.1 can be written in terms of the principal specialization of Schur functions.

Theorem 1.2. *For $0 < q < 1$, we have*

$$(5) \quad \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{k=1}^m \left(\prod_{i=1}^n x_i^{r_k} (qx_i; q)_{s_k} \prod_{1 \leq i < j \leq n} |x_j - x_i| \right) d_q x_1 \cdots d_q x_n \\ = (1-q)^n q^{(r+1)\binom{n}{2} + m\binom{n}{3}} \prod_{k=1}^m \prod_{h=s_k}^{n+s_k-1} (q; q)_h \sum_{\lambda \in \text{Par}_n} q^{|\lambda|} \prod_{k=1}^m q^{r_k |\lambda|} s_\lambda(1, q, \dots, q^{n+s_k-1}),$$

where Par_n is the set of all partitions of length $\leq n$. In particular, we have

$$(6) \quad \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i < j \leq n} |x_j - x_i|^m d_q x_1 \cdots d_q x_n \\ = (1-q)^n q^{(r+1)\binom{n}{2} + m\binom{n}{3}} \prod_{h=s}^{n+s-1} (q; q)_h \left(\prod_{h=1}^{n-1} (q; q)_h \right)^{m-1} \\ \times \sum_{\lambda \in \text{Par}_n} q^{(r+1)|\lambda|} s_\lambda(1, q, \dots, q^{n+s-1}) s_\lambda(1, q, \dots, q^{n-1})^{m-1}.$$

This allows us to evaluate the q -Selberg integral in Theorem 1.1 when $m = 2$ and $\mathbf{s} = (s, 0)$ (or $(0, s)$) (resp. $m = 1$ and $s = 0$ or 1), by using the Cauchy identity (resp. the Schur–Littlewood identity) for Schur functions.

The remainder of this paper is organized as follows. In Section 2 we define $(n, \mathbf{r}, \mathbf{s})$ -Young books. Considering $(n, \mathbf{r}, \mathbf{s})$ -Young books as linear extensions of a certain poset, we express the generating function for $(n, \mathbf{r}, \mathbf{s})$ -Young books according to the major index in terms of Schur functions. In Section 3 we prove Theorems 1.1 and 1.2 by expressing the generating function obtained in Section 2 as a Jackson integral. In Section 4 we evaluate special cases of the q -Selberg integral in Theorem 1.1 using Cauchy and Schur–Littlewood identities. In Section 5 we derive variants of the q -Selberg integral (2) using Cauchy-type identities for classical group characters.

2. THE GENERATING FUNCTION OF YOUNG BOOKS

In this section we define the Young books introduced in [6] and consider their generating function using the major index statistic of linear extensions of a poset.

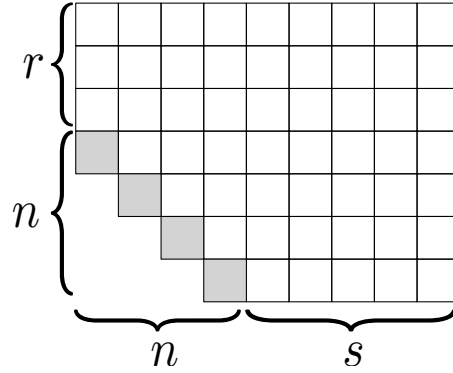


FIGURE 1. An (n, r, s) -staircase with $n = 4, r = 3$ and $s = 5$. The diagonal cells are shaded.

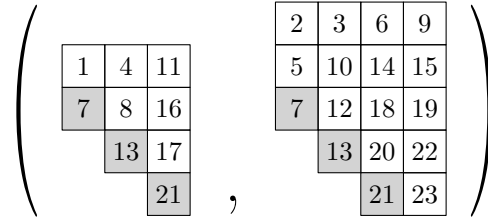


FIGURE 2. An $(n, \mathbf{r}, \mathbf{s})$ -Young book with $n = 3, \mathbf{r} = (1, 2)$, and $\mathbf{s} = (0, 1)$. The diagonal cells are shaded.

Throughout this paper we denote $[n] = \{1, 2, \dots, n\}$.

An (n, r, s) -staircase is the diagram obtained from an $(r + n) \times (n + s)$ rectangle by removing a staircase $(n - 1, n - 2, \dots, 1)$ from the lower-left corner, see Figure 1. We label the rows of an (n, r, s) -staircase by $-r + 1, -r + 2, \dots, -1, 0, 1, 2, \dots, n$ from top to bottom and the columns by $1, 2, \dots, n + s$ from left to right. For $1 \leq i \leq n$, the cell in the i th row and i th column is called the i th diagonal cell.

Let $\mathbf{r} = (r_1, r_2, \dots, r_m)$ and $\mathbf{s} = (s_1, s_2, \dots, s_m)$ be compositions of r and s respectively. In other words, $r_1 + r_2 + \dots + r_m = r$, $s_1 + s_2 + \dots + s_m = s$ and $r_i, s_i \geq 0$.

An $(n, \mathbf{r}, \mathbf{s})$ -staircase is an m -tuple $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ where each $\lambda^{(k)}$ is an (n, r_k, s_k) -staircase and the j th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ are identified for $j = 1, 2, \dots, n$. We say that $\lambda^{(k)}$ is the k th page of this $(n, \mathbf{r}, \mathbf{s})$ -staircase. Note that an $(n, \mathbf{r}, \mathbf{s})$ -staircase has N cells, where

$$N = m \binom{n}{2} + (r + s + 1)n + \sum_{k=1}^m r_k s_k.$$

An $(n, \mathbf{r}, \mathbf{s})$ -Young book is a filling of an $(n, \mathbf{r}, \mathbf{s})$ -staircase with integers $1, 2, \dots, N$ such that in each page the entries are increasing from left to right in each row and from top to bottom in each column. See Figure 2. We denote by $\text{YB}(n, \mathbf{r}, \mathbf{s})$ the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books.

For $B \in \text{YB}(n, \mathbf{r}, \mathbf{s})$, a descent of B is an integer i such that one of the following conditions holds:

- the row containing $i + 1$ has a smaller label than that of the row containing i regardless of their page numbers,
- i and $i + 1$ are non-diagonal entries and $i + 1$ appears in an earlier page than i with the same row label.

The major index $\text{maj}(B)$ of B is the sum of descents of B . For example, if B is the Young book in Figure 2, then $\text{maj}(B) = 1 + 5 + 8 + 10 + 13 + 17 + 21 = 75$.

We can consider an $(n, \mathbf{r}, \mathbf{s})$ -Young book as a linear extension of a poset. We briefly introduce some terminologies. We refer the reader to [14, 3.15] for details.

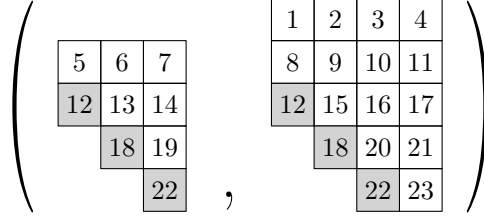


FIGURE 3. The $(n, \mathbf{r}, \mathbf{s})$ -Young book corresponding to the fixed linear extension ω of the poset $P_{n,\mathbf{r},\mathbf{s}}$ with $n = 3$, $\mathbf{r} = (1, 2)$, and $\mathbf{s} = (0, 1)$.

Let P be a poset with p elements. A *linear extension* of P is an order-preserving bijection $\sigma : P \rightarrow [p]$, i.e., if $x <_P y$ then $\sigma(x) < \sigma(y)$. We denote by $\mathcal{L}(P)$ the set of linear extensions of P . A *P-partition* is an order-reversing map $\sigma : P \rightarrow \mathbb{N}$, i.e., if $x <_P y$ then $\sigma(x) \geq \sigma(y)$. We denote by $\mathcal{A}(P)$ the set of all P -partitions.

Note that, once we fix a linear extension ω of P , we can consider $\pi \in \mathcal{L}(P)$ as a permutation on $[p]$ by $\omega\pi^{-1}$. For a permutation $\pi = \pi_1\pi_2 \dots \pi_p$, a *descent* of π is an integer $i \in [p-1]$ such that $\pi_i > \pi_{i+1}$. The *major index* $\text{maj}(\pi)$ is the sum of descents of π .

It is well known, for example [14, 3.15.7 Theorem], that for a poset P with a fixed linear extension $\omega : P \rightarrow [p]$, we have

$$(7) \quad \sum_{\sigma \in \mathcal{A}(P)} q^{|\sigma|} = \frac{\sum_{\pi \in \mathcal{L}(P)} q^{\text{maj}(\omega\pi^{-1})}}{(q; q)_p},$$

where $|\sigma| = \sum_{x \in P} \sigma(x)$.

Let $P_{n,\mathbf{r},\mathbf{s}}$ be the poset defined as follows:

- The underlying set consists of the cells in an $(n, \mathbf{r}, \mathbf{s})$ -staircase.
- The covering relation is defined by $c_1 < c_2$ if c_1 and c_2 are, respectively, in rows i and i' in columns j and j' in the same page such that $(i', j') = (i, j+1)$ or $(i', j') = (i+1, j)$. Here we assume that a diagonal cell is in every page.

Observe that $\text{YB}(n, \mathbf{r}, \mathbf{s})$ and $\mathcal{L}(P_{n,\mathbf{r},\mathbf{s}})$ are essentially the same: $B \in \text{YB}(n, \mathbf{r}, \mathbf{s})$ corresponds to $\pi \in \mathcal{L}(P_{n,\mathbf{r},\mathbf{s}})$ such that for a cell c , the label of c in B is $\pi(c)$.

Now we fix the linear extension $\omega : P_{n,\mathbf{r},\mathbf{s}} \rightarrow [N]$ defined uniquely by the following rules. Let c and c' be two cells in rows i and i' , columns j and j' , and pages k and k' , respectively, where we define $k = 0$ if c is a diagonal cell. Then we have $\omega(c) < \omega(c')$ if and only if one of the following conditions holds: (1) $i < i'$ or (2) $i = i'$ and $k < k'$ or (3) $i = i'$, $k = k'$, and $j < j'$, see Figure 3.

Note that if $B \in \text{YB}(n, \mathbf{r}, \mathbf{s})$ corresponds to $\pi \in \mathcal{L}(P_{n,\mathbf{r},\mathbf{s}})$, the descents of B and $\omega\pi^{-1}$ are the same. Thus we have

$$(8) \quad \sum_{B \in \text{YB}(n, \mathbf{r}, \mathbf{s})} q^{\text{maj}(B)} = \sum_{\pi \in \mathcal{L}(P_{n,\mathbf{r},\mathbf{s}})} q^{\text{maj}(\omega\pi^{-1})}.$$

By (7) and (8), we obtain

$$(9) \quad \sum_{\sigma \in \mathcal{A}(P_{n,\mathbf{r},\mathbf{s}})} q^{|\sigma|} = \frac{\sum_{B \in \text{YB}(n, \mathbf{r}, \mathbf{s})} q^{\text{maj}(B)}}{(q; q)_N}.$$

The *profile* of a $P_{n,\mathbf{r},\mathbf{s}}$ -partition $\sigma \in \mathcal{A}(P_{n,\mathbf{r},\mathbf{s}})$ is defined to be the partition obtained by reading the diagonal entries of σ . For a partition λ of length $\leq n$, let $\mathcal{A}_\lambda(P_{n,\mathbf{r},\mathbf{s}})$ be the set of all $P_{n,\mathbf{r},\mathbf{s}}$ -partitions with profile λ . Then we have

$$\sum_{\sigma \in \mathcal{A}(P_{n,\mathbf{r},\mathbf{s}})} q^{|\sigma|} = \sum_{\lambda \in \text{Par}_n} \sum_{\sigma \in \mathcal{A}_\lambda(P_{n,\mathbf{r},\mathbf{s}})} q^{|\sigma|}.$$

First we consider the case where $m = 1$. If a $P_{n+s,r,0}$ -partition has the profile $(\lambda_1, \dots, \lambda_n, 0, \dots, 0)$, where λ is a partition of length $\leq n$, then all the entries in the bottom triangle are 0, and the

remaining entries form a $P_{n,r,s}$ -partition with profile λ . So we have

$$\sum_{\sigma \in \mathcal{A}_{(\lambda_1, \dots, \lambda_n)}(P_{n,r,s})} q^{|\sigma|} = \sum_{\sigma \in \mathcal{A}_{(\lambda_1, \dots, \lambda_n, 0, \dots, 0)}(P_{n+s,r,0})} q^{|\sigma|}.$$

In general, the generating function of $P_{l,r,0}$ -partitions with fixed profile is expressed in terms of Schur function. The following proposition is a special case of [11, Theorem 2.1]. For the sake of completeness, we repeat the proof in our situation.

Proposition 2.1. *For a partition μ of length $\leq l$, we have*

$$(10) \quad \sum_{\sigma \in \mathcal{A}_\mu(P_{l,r,0})} q^{|\sigma|} = \frac{\prod_{h=1}^{r-1} (q; q)_h}{\prod_{h=l}^{l+r-1} (q; q)_h} s_\mu(q^{r+1}, q^{r+2}, \dots, q^{r+l}).$$

Proof. For $\sigma \in \mathcal{A}_\mu(P_{l,r,0})$, let $\sigma^{(i)}$ be the partition obtained by reading the entries in the cells in row k and column $k+i$ for some k from northwest to southeast. Then we have

$$\mu = \sigma^{(0)} \prec \sigma^{(1)} \prec \sigma^{(2)} \prec \dots \prec \sigma^{(r-1)} \succ \sigma^{(r)} \succ \sigma^{(r+1)} \succ \dots \succ \sigma^{(l+r)} = \emptyset,$$

where $\alpha \succ \beta$ means that the skew diagram α/β is a horizontal strip, i.e.,

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$$

Let Λ be the ring of symmetric functions, and define linear operators $H(t)$, $H^\perp(t)$, and $D(q)$ on Λ by

$$H(t)s_\alpha = \sum_{\gamma \succ \alpha} t^{|\gamma| - |\alpha|} s_\gamma, \quad H^\perp(t)s_\alpha = \sum_{\beta \prec \alpha} t^{|\alpha| - |\beta|} s_\beta, \quad D(q)s_\alpha = q^{|\alpha|} s_\alpha.$$

Note that $H(t)$ is the multiplication by $\sum_{k \geq 0} h_k t^k$. If we put

$$Z_\mu(q) = \sum_{\sigma \in \mathcal{A}_\mu(P_{l,r,0})} q^{|\sigma|},$$

then we have

$$\sum_{\mu \in \text{Par}_l} Z_\mu(q) s_\mu = (D(q)H^\perp(1))^r (D(q)H(1))^l s_\emptyset.$$

By using the relations

$$D(q)D(q') = D(qq'), \quad D(q)H(t) = H(qt)D(q), \quad \text{and} \quad D(q)H^\perp(t) = H^\perp(q^{-1}t)D(q),$$

we see that

$$\begin{aligned} & (D(q)H^\perp(1))^r (D(q)H(1))^l \\ &= H^\perp(q^{-1})H^\perp(q^{-2}) \dots H^\perp(q^{-r})H(q^{r+1})H(q^{r+2}) \dots H(q^{r+l})D(q^{r+l}). \end{aligned}$$

Next, by applying the commuting relation [10, I.5 Example 29]

$$H^\perp(s)H(t) = \frac{1}{1-st} H(t)H^\perp(s),$$

we obtain

$$\begin{aligned} & H^\perp(q^{-1})H^\perp(q^{-2}) \dots H^\perp(q^{-r})H(q^{r+1})H(q^{r+2}) \dots H(q^{r+l}) \\ &= \prod_{i=1}^r \prod_{j=1}^l \frac{1}{1 - q^{(-i)+(r+j)}} H(q^{r+1})H(q^{r+2}) \dots H(q^{r+l})H^\perp(q^{-1})H^\perp(q^{-2}) \dots H^\perp(q^{-r}). \end{aligned}$$

Since $D(q)s_\emptyset = s_\emptyset$ and $H^\perp(t)s_\emptyset = s_\emptyset$, we have

$$\sum_{\mu \in \text{Par}_l} Z_\mu(q) s_\mu = \prod_{i=1}^r \prod_{j=1}^l \frac{1}{1 - q^{(-i)+(r+j)}} H(q^{r+1})H(q^{r+2}) \dots H(q^{r+l})s_\emptyset.$$

Lastly, by appealing to the Cauchy identity

$$H(x_1)H(x_2) \dots H(x_l)s_\emptyset = \sum_{\mu \in \text{Par}_l} s_\mu(x_1, \dots, x_l)s_\mu,$$

we conclude that

$$\sum_{\mu \in \text{Par}_l} Z_\mu(q) s_\mu = \frac{\prod_{h=1}^{r-1} (q; q)_h}{\prod_{h=l}^{l+r-1} (q; q)_h} \sum_{\mu \in \text{Par}_l} s_\mu(q^{r+1}, \dots, q^{r+l}) s_\mu.$$

Equating the coefficient of s_μ completes the proof. \square

Now we can compute the generating function of $P_{n,r,s}$ -partitions.

Proposition 2.2. *Let $\mathbf{r} = (r_1, \dots, r_m)$ and $\mathbf{s} = (s_1, \dots, s_m)$ be compositions of r and s respectively. Then we have*

$$(11) \quad \sum_{\sigma \in \mathcal{A}(P_{n,r,s})} q^{|\sigma|} = \prod_{k=1}^m \frac{\prod_{h=1}^{r_k-1} (q; q)_h}{\prod_{h=n+s_k}^{n+r_k+s_k-1} (q; q)_h} \sum_{\lambda \in \text{Par}_n} q^{|\lambda|} \prod_{k=1}^m q^{r_k|\lambda|} s_\lambda(1, q, \dots, q^{n+s_k-1}).$$

Proof. For a $P_{n,r,s}$ -partition σ with profile λ , let σ_k be the P_{n,r_k,s_k} -partition on the k th page. Then we have

$$|\sigma| = |\lambda| + \sum_{k=1}^m (|\sigma_k| - |\lambda|).$$

Hence we have

$$\sum_{\sigma \in \mathcal{A}(P_{n,r,s})} q^{|\sigma|} = \sum_{\lambda \in \text{Par}_n} q^{|\lambda|} \prod_{k=1}^m \sum_{\sigma_k \in \mathcal{A}_\lambda(P_{n,r_k,s_k})} q^{|\sigma_k| - |\lambda|}.$$

On the other hand, it follows from Proposition 2.1 and the homogeneity of Schur function that

$$\begin{aligned} \sum_{\sigma_k \in \mathcal{A}_\lambda(P_{n,r_k,s_k})} q^{|\sigma_k| - |\lambda|} &= q^{-|\lambda|} \frac{\prod_{h=1}^{r_k-1} (q; q)_h}{\prod_{h=n+s_k}^{n+r_k+s_k-1} (q; q)_h} s_\lambda(q^{r_k+1}, q^{r_k+2}, \dots, q^{n+r_k+s_k}) \\ &= q^{r_k|\lambda|} \frac{\prod_{h=1}^{r_k-1} (q; q)_h}{\prod_{h=n+s_k}^{n+r_k+s_k-1} (q; q)_h} s_\lambda(1, q, \dots, q^{n+s_k-1}). \end{aligned}$$

\square

3. SCHUR FUNCTIONS AND q -SELBERG INTEGRAL

In this section, we prove Theorem 1.2, from which Theorem 1.1 follows.

We begin with a general formula for the Jackson integral.

Proposition 3.1. *Let f be a function in x_1, \dots, x_n satisfying the following two conditions:*

- (a) *f is symmetric in x_1, \dots, x_n .*
- (b) *If $x_i = x_j$ for $i \neq j$, then $f(x_1, \dots, x_n) = 0$.*

Then we have

$$(12) \quad \int_{[0,1]^n} f(x) d_q x = n! (1-q)^n \sum_{\lambda \in \text{Par}_n} q^{|\lambda| + \binom{n}{2}} f(q^{\lambda_1+n-1}, q^{\lambda_2+n-2}, \dots, q^{\lambda_n}).$$

Proof. We put

$$S = \int_{[0,1]^n} f(x) d_q x.$$

By definition, we have

$$S = (1-q)^n \sum_{k_1, \dots, k_n \geq 0} f(q^{k_1}, \dots, q^{k_n}) q^{k_1 + \dots + k_n}.$$

Condition (b) implies that $f(q^{k_1}, \dots, q^{k_n}) = 0$ if $k_i = k_j$ for some $i \neq j$. Hence we may assume that k_1, \dots, k_n are distinct in the summation. For a permutation $\sigma \in \mathfrak{S}_n$, we define

$$S_\sigma = (1-q)^n \sum_{k_{\sigma(1)} > \dots > k_{\sigma(n)} \geq 0} f(q^{k_1}, \dots, q^{k_n}) q^{k_1 + \dots + k_n}.$$

Then we have

$$S = \sum_{\sigma \in \mathfrak{S}_n} S_\sigma,$$

and Condition (a) implies that

$$S_\sigma = S_e \quad (\sigma \in \mathfrak{S}_n),$$

where e is the identity permutation. Therefore we have

$$S = n!S_e = n!(1-q)^n \sum_{k_1 > \dots > k_n \geq 0} f(q^{k_1}, \dots, q^{k_n}) q^{k_1 + \dots + k_n}.$$

For $k_1 > \dots > k_n \geq 0$, let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the partition given by $k_i = \lambda_i + (n-i)$. Then we have

$$S = n!(1-q)^n \sum_{\lambda \in \text{Par}_n} q^{|\lambda| + \binom{n}{2}} f(q^{\lambda_1 + n-1}, \dots, q^{\lambda_n}).$$

□

The principal specialization of Schur function can be written in the following form.

Lemma 3.2. *Let n be a positive integer and λ a partition of length $\leq n$. If s is a nonnegative integer, then we have*

$$(13) \quad s_\lambda(1, q, \dots, q^{n+s-1}) = \frac{q^{-\binom{n}{3}}}{\prod_{h=s}^{n+s-1} (q; q)_h} \prod_{i=1}^n (q \cdot q^{\lambda_i + n-i}; q)_s \prod_{1 \leq i < j \leq n} (q^{\lambda_j + n-j} - q^{\lambda_i + n-i}).$$

Proof. Using the Vandermonde determinant, we have

$$s_\lambda(1, q, \dots, q^{n+s-1}) = \frac{\prod_{1 \leq i < j \leq n+s} (q^{\lambda_j + n+s-j} - q^{\lambda_i + n+s-i})}{\prod_{1 \leq i < j \leq n+s} (q^{n+s-j} - q^{n+s-i})}.$$

Since $\lambda_{n+1} = \dots = \lambda_{n+s} = 0$, we have

$$\begin{aligned} \prod_{1 \leq i < j \leq n} (q^{\lambda_j + n+s-j} - q^{\lambda_i + n+s-i}) &= q^{s\binom{n}{2}} \prod_{1 \leq i < j \leq n} (q^{\lambda_j + n-j} - q^{\lambda_i + n-i}), \\ \prod_{i=1}^n \prod_{j=n+1}^{n+s} (q^{\lambda_j + n+s-j} - q^{\lambda_i + n+s-i}) &= q^{n\binom{s}{2}} \prod_{i=1}^n (q^{\lambda_i + n-i+1}; q)_s, \\ \prod_{n+1 \leq i < j \leq n+s} (q^{\lambda_j + n+s-j} - q^{\lambda_i + n+s-i}) &= q^{\binom{s}{3}} \prod_{h=1}^{s-1} (q; q)_h. \end{aligned}$$

On the other hand, the denominator is equal to

$$\prod_{1 \leq i < j \leq n+s} (q^{n+s-j} - q^{n+s-i}) = q^{\binom{n+s}{3}} \prod_{h=1}^{n+s-1} (q; q)_h.$$

We can complete the proof by noting the relation

$$s\binom{n}{2} + n\binom{s}{2} + \binom{s}{3} - \binom{n+s}{3} = -\binom{n}{3}.$$

□

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. We put

$$f_{n,r,s}(x_1, \dots, x_n) = \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i < j \leq n} |x_j - x_i|.$$

Then, by using Proposition 3.1, we have

$$\frac{1}{n!} \int_{[0,1]^n} \prod_{k=1}^m f_{n,r_k,s_k}(x_1, \dots, x_n) d_q x = (1-q)^n \sum_{\lambda \in \text{Par}_n} q^{|\lambda| + \binom{n}{2}} \prod_{k=1}^m f(q^{\lambda_1 + n-1}, \dots, q^{\lambda_n}).$$

It follows from Lemma 3.2 that

$$\begin{aligned}
f_{n,r,s}(q^{\lambda_1+n-1}, \dots, q^{\lambda_n}) &= \prod_{i=1}^n (q^{\lambda_i+n-i})^r (q \cdot q^{\lambda_i+n-i}; q)_s \prod_{1 \leq i < j \leq n} |q^{\lambda_j+n-j} - q^{\lambda_i+n-i}| \\
&= q^{r|\lambda|+r\binom{n}{2}} \prod_{i=1}^n (q \cdot q^{\lambda_i+n-i}; q)_s \prod_{1 \leq i < j \leq n} (q^{\lambda_j+n-j} - q^{\lambda_i+n-i}) \\
&= q^{r|\lambda|+r\binom{n}{2}+\binom{n}{3}} \prod_{h=s}^{n+s-1} (q; q)_h s_\lambda(1, q, \dots, q^{n+s-1}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\frac{1}{n!} \int_{[0,1]^n} \prod_{k=1}^m f_{n,r_k,s_k}(x_1, \dots, x_n) d_q x \\
&= (1-q)^n \sum_{\lambda \in \text{Par}_n} q^{|\lambda|+\binom{n}{2}} \prod_{k=1}^m q^{r_k|\lambda|+r_k\binom{n}{2}+\binom{n}{3}} \prod_{h=s_k}^{n+s_k-1} (q; q)_h s_\lambda(1, q, \dots, q^{n+s_k-1}) \\
&= q^{(r+1)\binom{n}{2}+m\binom{n}{3}} (1-q)^n \prod_{k=1}^m \prod_{h=s_k}^{n+s_k-1} (q; q)_h \sum_{\lambda \in \text{Par}_n} q^{|\lambda|} \prod_{k=1}^m q^{r_k|\lambda|} s_\lambda(1, q, \dots, q^{n+s_k-1}).
\end{aligned}$$

□

By Proposition 2.2, Theorem 1.2 and (7), we obtain Theorem 1.1.

4. EVALUATION OF q -SELBERG INTEGRALS

In this section, we use the Cauchy and Schur–Littlewood identities to evaluate some special cases of the q -Selberg integral in Theorem 1.1.

Recall the Cauchy and Schur–Littlewood identities:

$$(14) \quad \sum_{\lambda \in \text{Par}} s_\lambda(x) s_\lambda(y) = \frac{1}{\prod_{i,j} (1 - x_i y_j)},$$

$$(15) \quad \sum_{\lambda \in \text{Par}} s_\lambda(x) = \frac{1}{\prod_i (1 - x_i) \prod_{i < j} (1 - x_i x_j)},$$

where Par denotes the set of all partitions.

Theorem 4.1. *For $0 < q < 1$, we have*

$$\begin{aligned}
(16) \quad &\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r \prod_{1 \leq i < j \leq n} |x_j - x_i| d_q x_1 \cdots d_q x_n \\
&= n! q^{(r+1)\binom{n}{2}+\binom{n}{3}} \frac{[1]_q! [2]_q! \cdots [n-1]_q!}{\prod_{i=1}^n [r+i]_q \prod_{1 \leq i < j \leq n} [2r+i+j]_q},
\end{aligned}$$

$$\begin{aligned}
(17) \quad &\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1 - qx_i) \prod_{1 \leq i < j \leq n} |x_j - x_i| d_q x_1 \cdots d_q x_n \\
&= n! q^{(r+1)\binom{n}{2}+\binom{n}{3}} \frac{[1]_q! [2]_q! \cdots [n]_q! [(n+1)r + \binom{n+1}{2}]_q}{\prod_{i=1}^{n+1} [r+i]_q \prod_{1 \leq i < j \leq n+1} [2r+i+j]_q},
\end{aligned}$$

$$\begin{aligned}
(18) \quad &\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x_1 \cdots d_q x_n \\
&= n! q^{(r+1)\binom{n}{2}+2\binom{n}{3}} \frac{[1]_q! [2]_q! \cdots [n-1]_q! [s+1]_q! [s+2]_q! \cdots [s+n-1]_q!}{\prod_{i=1}^{n+s} \prod_{j=1}^n [r+i+j-1]_q}.
\end{aligned}$$

Remark 4.1. The last formula (18) is equivalent to the $k = 1$ case of (2). See [4, Section 4].

Proof. By applying (6) with $s = 0$ and $m = 1$ and using the homogeneity of Schur function, we have

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r \prod_{1 \leq i < j \leq n} |x_j - x_i| d_q x_1 \cdots d_q x_n \\ = n!(1-q)^n q^{(r+1)\binom{n}{2} + \binom{n}{3}} \prod_{h=1}^{n-1} (q; q)_h \sum_{\lambda \in \text{Par}_n} s_\lambda(q^{r+1}, q^{r+2}, \dots, q^{r+n}). \end{aligned}$$

Thus (16) is obtained by specializing $x_i = q^{r+i}$ for $1 \leq i \leq n$ and $x_i = 0$ for $i > n$ in the Schur–Littlewood identity (15).

Next we apply (6) with $s = 1$ and $m = 1$. Then we have

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1 - qx_i) \prod_{1 \leq i < j \leq n} |x_j - x_i| d_q x_1 \cdots d_q x_n \\ = n!(1-q)^n q^{(r+1)\binom{n}{2} + \binom{n}{3}} \prod_{h=1}^n (q; q)_h \sum_{\lambda \in \text{Par}_n} s_\lambda(q^{r+1}, q^{r+2}, \dots, q^{r+n+1}). \end{aligned}$$

For $\lambda \in \text{Par}_{n+1}$ and $\lambda + (1^{n+1}) = (\lambda_1 + 1, \dots, \lambda_{n+1} + 1)$, we have

$$s_{\lambda + (1^{n+1})}(x_1, \dots, x_{n+1}) = x_1 \cdots x_{n+1} s_\lambda(x_1, \dots, x_{n+1}).$$

Hence, by using (15), we see that

$$\begin{aligned} \sum_{\lambda \in \text{Par}_n} s_\lambda(x_1, \dots, x_{n+1}) &= \sum_{\lambda \in \text{Par}_{n+1}} s_\lambda(x_1, \dots, x_{n+1}) - \sum_{\lambda \in \text{Par}_{n+1}, \lambda_{n+1} > 0} s_\lambda(x_1, \dots, x_{n+1}) \\ &= \frac{1 - x_1 \cdots x_{n+1}}{\prod_{i=1}^{n+1} (1 - x_i) \prod_{1 \leq i < j \leq n+1} (1 - x_i x_j)}. \end{aligned}$$

Thus (17) is obtained by specializing $x_i = q^{r+i}$ for $1 \leq i \leq n+1$.

Finally we use (6) in the case $m = 2$. Then we have

$$\begin{aligned} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x_1 \cdots d_q x_n \\ = n!(1-q)^n q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{h=s}^{n+s-1} (q; q)_h \prod_{h=1}^{n-1} (q; q)_h \sum_{\lambda \in \text{Par}_n} s_\lambda(1, q, \dots, q^{n+s-1}) s_\lambda(q^{r+1}, q^{r+2}, \dots, q^{r+n}). \end{aligned}$$

Hence (18) follows from the Cauchy identity (14). \square

Remark 4.2. We have product formulas for the q -Selberg integral in Theorem 1.1 in the case $m = 1$ and $s \in \{0, 1\}$. If $m = 1$ and $s \geq 2$, then we can evaluate the q -Selberg integral by using a column-length restricted Schur–Littlewood formula [8]

$$\sum_{\lambda \in \text{Par}_n} s_\lambda(x) = \frac{1}{\prod_i (1 - x_i) \prod_{i < j} (1 - x_i x_j)} \cdot \sum_{\mu} (-1)^{(|\mu| - (n-1)p(\mu))/2} s_\mu,$$

where μ runs over all partitions of the form $\mu = (\alpha | \alpha + n)$ in the Frobinius notation, and $p(\mu)$ is the largest integer k such that μ contains a $k \times k$ square. However we do not have simple product formulas for $s \geq 2$.

5. VARIANTS OF THE q -SELBERG INTEGRAL

Under the same idea as in the previous section, we use the Cauchy-type identities for classical group characters (see [7] and [9]) to derive variants of the q -Selberg integral (2).

Theorem 5.1. *Let n be a positive integer and s be a nonnegative integer. Then we have*

$$(19) \quad \int_{[0,1]^n} \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i < j \leq n} (1 - q^{s+1} x_i x_j) \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x$$

$$= n! q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^n [s + 2k - 2]_q! \frac{\prod_{1 \leq i < j \leq n} [2n + 2r + s + i + j - 2]_q}{\prod_{i=1}^{2n+s-1} \prod_{j=1}^n [r + i + j - 1]_q},$$

$$(20) \quad \int_{[0,1]^n} \prod_{i=1}^n x_i^r (qx_i; q)_s (1 + q^{(s+1)/2} x_i) \prod_{1 \leq i < j \leq n} (1 - q^{s+1} x_i x_j) \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x$$

$$= n! q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^n [s + 2k - 2]_q! \prod_{k=1}^n (1 + q^{s/2+k-1/2})$$

$$\times \frac{\prod_{i=1}^n [n + r + s/2 + i - 1/2]_q \prod_{1 \leq i < j \leq n} [2n + 2r + s + i + j - 1]_q}{\prod_{i=1}^{2n+s} \prod_{j=1}^n [r + i + j - 1]_q},$$

$$(21) \quad \int_{[0,1]^n} \prod_{i=1}^n x_i^r (qx_i; q)_s (1 - q^{(s+1)/2} x_i) \prod_{1 \leq i < j \leq n} (1 - q^{s+1} x_i x_j) \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x$$

$$= n! q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^n [s + 2k - 2]_q! \prod_{k=1}^n [s/2 + k - 1/2]_q$$

$$\times \frac{\prod_{i=1}^n (1 + q^{n+r+s/2+i-1/2}) \prod_{1 \leq i < j \leq n} [2n + 2r + s + i + j - 1]_q}{\prod_{i=1}^{2n+s} \prod_{j=1}^n [r + i + j - 1]_q},$$

$$(22) \quad \int_{[0,1]^n} \prod_{i=1}^n x_i^r (qx_i; q)_s \prod_{1 \leq i \leq j \leq n} (1 - q^{s+1} x_i x_j) \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x$$

$$= n! q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^n [s + 2k - 1]_q! \frac{\prod_{1 \leq i \leq j \leq n} [2n + 2r + s + i + j]_q}{\prod_{i=1}^{2n+s+1} \prod_{j=1}^n [r + i + j - 1]_q}.$$

In order to prove this theorem, we use the classical group characters. Let $\lambda \in \text{Par}_N$ be a partition of length $\leq N$. The *symplectic Schur function* $s_{\langle \lambda \rangle}^C(x_1, \dots, x_N)$, and the *orthogonal*

Schur function $s_{[\lambda]}^B(x_1, \dots, x_N)$, $s_{[\lambda]}^D(x_1, \dots, x_N)$ are defined by putting

$$(23) \quad s_{\langle \lambda \rangle}^C(x_1, \dots, x_N) = \frac{\det \left(x_i^{\lambda_j + N - j + 1} - x_i^{-(\lambda_j + N - j + 1)} \right)_{1 \leq i, j \leq N}}{\det \left(x_i^{N - j + 1} - x_i^{-(N - j + 1)} \right)_{1 \leq i, j \leq N}},$$

$$(24) \quad s_{[\lambda]}^B(x_1, \dots, x_N) = \frac{\det \left(x_i^{\lambda_j + N - j + 1/2} - x_i^{-(\lambda_j + N - j + 1/2)} \right)_{1 \leq i, j \leq N}}{\det \left(x_i^{N - j + 1/2} - x_i^{-(N - j + 1/2)} \right)_{1 \leq i, j \leq N}},$$

$$(25) \quad s_{\langle \lambda \rangle}^D(x_1, \dots, x_N) = \begin{cases} \frac{\det \left(x_i^{\lambda_j + N - j} + x_i^{-(\lambda_j + N - j)} \right)_{1 \leq i, j \leq N}}{\frac{1}{2} \det \left(x_i^{N - j} + x_i^{-(N - j)} \right)_{1 \leq i, j \leq N}} & \text{if } \lambda_N > 0, \\ \frac{\det \left(x_i^{\lambda_j + N - j} + x_i^{-(\lambda_j + N - j)} \right)_{1 \leq i, j \leq N}}{\det \left(x_i^{N - j} + x_i^{-(N - j)} \right)_{1 \leq i, j \leq N}} & \text{if } \lambda_N = 0. \end{cases}$$

For a decreasing sequence of the form $\lambda + 1/2 = (\lambda_1 + 1/2, \dots, \lambda_N + 1/2)$, where $\lambda \in \text{Par}_N$, we define the *spinor Schur function* $s_{[\lambda+1/2]}^B(x_1, \dots, x_N)$ and $s_{[\lambda+1/2]}^D(x_1, \dots, x_N)$ by putting

$$(26) \quad s_{[\lambda+1/2]}^B(x_1, \dots, x_N) = \frac{\det \left(x_i^{\lambda_j + N - j + 1} - x_i^{-(\lambda_j + N - j + 1)} \right)_{1 \leq i, j \leq N}}{\det \left(x_i^{N - j + 1/2} - x_i^{-(N - j + 1/2)} \right)_{1 \leq i, j \leq N}},$$

$$(27) \quad s_{[\lambda+1/2]}^D(x_1, \dots, x_N) = \frac{\det \left(x_i^{\lambda_j + N - j + 1/2} + x_i^{-(\lambda_j + N - j + 1/2)} \right)_{1 \leq i, j \leq N}}{\frac{1}{2} \det \left(x_i^{N - j} + x_i^{-(N - j)} \right)_{1 \leq i, j \leq N}}.$$

These schur functions are irreducible characters of classical groups. Note that

$$s_{[\lambda+1/2]}^B(x_1, \dots, x_N) = \prod_{i=1}^N (x_i^{1/2} + x_i^{-1/2}) s_{\langle \lambda \rangle}^C(x_1, \dots, x_N).$$

Then we have the following Cauchy-type identities, which can be proved by using the Cauchy–Binet formula.

Proposition 5.2. *Let N and n be positive integers satisfying $N \geq n$. Then we have*

(28)

$$\sum_{\lambda \in \text{Par}_n} s_{\langle \lambda \rangle}^C(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) = \frac{\prod_{1 \leq i < j \leq n} (1 - u_i u_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) (1 - x_i^{-1} u_j)},$$

(29)

$$\sum_{\lambda \in \text{Par}_n} s_{[\lambda]}^B(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) = \frac{\prod_{i=1}^n (1 + u_i) \prod_{1 \leq i < j \leq n} (1 - u_i u_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) (1 - x_i^{-1} u_j)},$$

(30)

$$\sum_{\lambda \in \text{Par}_n} s_{[\lambda+1/2]}^B(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) = \frac{\prod_{i=1}^n (x_i^{1/2} + x_i^{-1/2}) \prod_{1 \leq i < j \leq n} (1 - u_i u_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) (1 - x_i^{-1} u_j)},$$

(31)

$$\sum_{\lambda \in \text{Par}_n} s_{[\lambda]}^D(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) = \frac{\prod_{1 \leq i \leq j \leq n} (1 - u_i u_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) (1 - x_i^{-1} u_j)},$$

(32)

$$\sum_{\lambda \in \text{Par}_n} s_{[\lambda+1/2]}^D(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) = \frac{\prod_{i=1}^n (x_i^{1/2} + x_i^{-1/2}) \prod_{i=1}^n (1 - u_i) \prod_{1 \leq i < j \leq n} (1 - u_i u_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) (1 - x_i^{-1} u_j)}.$$

Proof of Theorem 5.1. Since the arguments are the same, we illustrate how to prove (19) in the case s is odd. For other cases, see Table 1.

TABLE 1. Proof of Theorem 5.1

Identity	Parity of s	Character	Specialization
(19)	odd	$s_{\langle \lambda \rangle}^C$	$x_i = q^{N-i+1/2}$
(19)	even	$s_{[\lambda+1/2]}^D$	$x_i = q^{N-i}$
(20)	odd	$s_{\langle \lambda \rangle}^C$	$x_i = q^{N-i+1}$
(20)	even	$s_{[\lambda+1/2]}^D$	$x_i = q^{N-i+1/2}$
(21)	odd	$s_{[\lambda]}^D$	$x_i = q^{N-i}$
(21)	even	$s_{[\lambda]}^B$	$x_i = q^{N-i+1/2}$
(22)	odd	$s_{[\lambda]}^D$	$x_i = q^{N-i+1/2}$
(22)	even	$s_{[\lambda]}^B$	$x_i = q^{N-i+1}$

Suppose that $s = 2l + 1$ is odd and put $N = n + l$. It follows from the Weyl denominator formula (or the Vandermonde determinant) that, for a partition λ of length $\leq N$,

$$\begin{aligned} & s_{\langle \lambda \rangle}^C(q^{N-1/2}, q^{N-3/2}, \dots, q^{3/2}, q^{1/2}) \\ &= q^{-(N-1/2)|\lambda|} \prod_{i=1}^N \frac{1 - q^{\lambda_i + N - i + 1}}{1 - q^{N - i + 1}} \prod_{1 \leq i < j \leq N} \frac{q^{\lambda_j + N - j + 1} - q^{\lambda_i + N - i + 1}}{q^{N - j + 1} - q^{N - i + 1}} \frac{1 - q^{\lambda_i + N - i + 1}}{1 - q^{N - i + 1}} \cdot \frac{q^{\lambda_j + N - j + 1}}{q^{N - j + 1}}. \end{aligned}$$

If $l(\lambda) \leq n$, then we have

$$s_{\langle \lambda \rangle}^C(q^{N-1/2}, q^{N-3/2}, \dots, q^{3/2}, q^{1/2}) = q^{-(N-1/2)|\lambda| - \binom{n}{3}} \frac{1}{\prod_{k=1}^n (q; q)_{s+2k-2}} \cdot F(q^{\lambda_1 + n - 1}, \dots, q^{\lambda_n}),$$

where

$$F(x_1, \dots, x_n) = \prod_{i=1}^n (qx_i; q)_s \prod_{1 \leq i < j \leq n} (1 - q^{s+1} x_i x_j) \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Hence, by applying Proposition 3.1, we see that

$$\begin{aligned}
& \int_{[0,1]^n} \prod_{i=1}^n x_i^r(qx_i; q)_s \prod_{1 \leq i < j \leq n} (1 - q^{s+1}x_i x_j) \prod_{1 \leq i < j \leq n} |x_j - x_i|^2 d_q x \\
&= n!(1-q)^n \sum_{\lambda \in \text{Par}_n} q^{|\lambda| + \binom{n}{2}} \prod_{i=1}^n (q^{\lambda_i + n - i})^r F(q^{\lambda_1 + n - 1}, \dots, q^{\lambda_n}) \prod_{1 \leq i < j \leq n} (q^{\lambda_j + n - j} - q^{\lambda_i + n - i}) \\
&= n!(1-q)^n q^{(r+1)\binom{n}{2} + 2\binom{n}{3}} \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^n [s + 2k - 2]_q! \\
&\quad \times \sum_{\lambda \in \text{Par}_n} s_{\langle \lambda \rangle}^C(q^{N-1/2}, q^{N-3/2}, \dots, q^{3/2}, q^{1/2}) \cdot q^{(r+N+1/2)|\lambda|} s_{\lambda}(1, q, \dots, q^{n-1})
\end{aligned}$$

Now, by specializing $x_i = q^{N-i+1/2}$ and $u_j = q^{r+N+j-1/2}$ in the Cauchy-type identity (28), we obtain (19). \square

Similarly, by using the Cauchy-type formula for Schur functions corresponding to rational irreducible representations of the general linear group, we can derive

Theorem 5.3. *Let n, m and l be nonnegative integers with $N = n + m + l > 0$. Then we have*

$$\begin{aligned}
(33) \quad & \int_{[0,1]^{n+m}} \prod_{i=1}^n x_i^r(qx_i; q)_l \prod_{j=1}^m y_j^s(qy_j; q)_l \prod_{i=1}^n \prod_{j=1}^m (1 - q^l x_i y_j) \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 \prod_{1 \leq i < j \leq m} (y_j - y_i)^2 d_q x d_q y \\
&= n!m!q^{(r+1)\binom{n}{2} + (s+1)\binom{m}{2} + 2\binom{n}{3} + 2\binom{m}{3}} \frac{\prod_{k=1}^{N-1} [k]_q! \prod_{k=1}^{n-1} [k]_q! \prod_{k=1}^{m-1} [k]_q!}{\prod_{k=1}^{l-1} [k]_q!} \\
&\quad \times \frac{\prod_{i=1}^n \prod_{j=1}^m [N + r + s + i + j - 1]_q}{\prod_{i=1}^n \prod_{k=1}^N [r + i + k - 1]_q \prod_{j=1}^m \prod_{k=1}^N [s + j + k - 1]_q}.
\end{aligned}$$

In order to prove this theorem, we use the generalized Schur function defined by

$$s_{\Lambda}(x_1, \dots, x_N) = \frac{\det \left(x_i^{\Lambda_j + N - j} \right)_{1 \leq i, j \leq N}}{\det \left(x_i^{N - j} \right)_{1 \leq i, j \leq N}},$$

where $\Lambda \in \mathbb{Z}^n$ is a decreasing sequence of integers. For two partitions λ and μ with $l(\lambda) + l(\mu) \leq N$, we define a weakly decreasing sequence $\Lambda(\lambda, \mu) \in \mathbb{Z}^N$ by

$$\Lambda(\lambda, \mu) = (\lambda_1, \dots, \lambda_{l(\lambda)}, 0, \dots, 0, -\mu_{l(\mu)}, \dots, -\mu_1).$$

Then we have the following Cauchy-type formula:

Proposition 5.4. *If $N \geq n + m$, then we have*

$$\begin{aligned}
(34) \quad & \sum_{\lambda \in \text{Par}_n, \mu \in \text{Par}_m} s_{\Lambda(\lambda, \mu)}(x_1, \dots, x_N) s_{\lambda}(u_1, \dots, u_n) s_{\mu}(v_1, \dots, v_m) \\
&= \frac{\prod_{i=1}^n \prod_{j=1}^m (1 - u_i v_j)}{\prod_{i=1}^N \prod_{j=1}^n (1 - x_i u_j) \prod_{j=1}^m (1 - x_i^{-1} v_j)}.
\end{aligned}$$

Also we need the following extension of Proposition 3.1.

Proposition 5.5. *Let g be a function in $x_1, \dots, x_n, y_1, \dots, y_m$ satisfying the following four conditions:*

- (a1) g is symmetric in x_1, \dots, x_n ;
- (a2) g is symmetric in y_1, \dots, y_m ;
- (b1) If $x_i = x_j$ for $i \neq j$, then $g(x_1, \dots, x_n, y_1, \dots, y_m) = 0$;
- (b2) If $y_i = y_j$ for $i \neq j$, then $g(x_1, \dots, x_n, y_1, \dots, y_m) = 0$.

Then we have

$$(35) \quad \int_{[0,1]^{n+m}} g(x, y) d_q x d_q y \\ = (1-q)^{n+m} n! m! \sum_{\lambda \in \text{Par}_n, \mu \in \text{Par}_m} q^{|\lambda|+|\mu|+\binom{n}{2}+\binom{m}{2}} g(q^{\lambda_1+n-1}, q^{\lambda_2+n-2}, \dots, q^{\lambda_n}, q^{\mu_1+m-1}, q^{\mu_2+m-2}, \dots, q^{\mu_m}).$$

Proof of Theorem 5.3. The idea of the proof is similar to that of Theorem 5.1.

If we put

$$G(x_1, \dots, x_n, y_1, \dots, y_m) \\ = \prod_{i=1}^n (qx_i; q)_l \prod_{j=1}^m (qy_j; q)_l \prod_{i=1}^n \prod_{j=1}^m (1 - q^l x_i y_j) \prod_{1 \leq i < j \leq n} (x_j - x_i) \prod_{1 \leq i < j \leq m} (y_j - y_i),$$

then we have

$$s_{\Lambda(\lambda, \mu)}(1, q, \dots, q^{n-1}) = q^{-(N-1)|\mu| - \binom{n}{3} - \binom{m}{3}} \frac{\prod_{h=1}^{l-1} (q; q)_h}{\prod_{h=1}^{N-1} (q; q)_h} \cdot G(q^{\lambda_1+n-1}, \dots, q^{\lambda_n}).$$

Hence, by applying Proposition 5.5, we see that

$$\int_{[0,1]^{n+m}} \prod_{i=1}^n x_i^T (qx_i; q)_l \prod_{j=1}^m y_j^S (qy_j; q)_l \prod_{i=1}^n \prod_{j=1}^m (1 - q^l x_i y_j) \prod_{1 \leq i < j \leq n} (x_j - x_i)^2 \prod_{1 \leq i < j \leq m} (y_j - y_i)^2 d_q x d_q y \\ = n! m! (1-q)^{n+m} q^{(r+1)\binom{n}{2} + (s+1)\binom{m}{2} + 2\binom{n}{3} + 2\binom{m}{3}} \frac{\prod_{h=1}^{N-1} (q; q)_h \prod_{h=1}^{n-1} (q; q)_h \prod_{h=1}^{m-1} (q; q)_h}{\prod_{h=1}^{l-1} (q; q)_h} \\ \times \sum_{\lambda \in \text{Par}_n, \mu \in \text{Par}_m} s_{\Lambda(\lambda, \mu)}(1, q, \dots, q^{N-1}) \cdot q^{(r+1)|\lambda|} s_{\lambda}(1, q, \dots, q^{n-1}) \cdot q^{(s+N)|\mu|} s_{\mu}(1, q, \dots, q^{m-1}).$$

Now we can complete the proof by using the Cauchy-type identity (34). \square

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